Conservation of Energy–Momentum in Teleparallel Gravity.

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Abstract

In a well-known paper [1] V.C. de Andrade, L. C. T. Guillen and J.G. Pereira defined a conserved gauge current $hj_a{}^\rho$, however they stated that: "*This is, we believe the farthest one can go in the direction of a tensorial definition for the energy and momentum of the gravitational field. The lack of local Lorentz covariance can be considered as the teleparallel manifestation of the pseudotensor character of the gravitational energy–momentum density in general relativity...*". Well, we believe that they stopped just less than an inch before giving such a tensorial definition, and furthermore that the resulting energy–momentum tensor has zero trace and can be made symmetric, as a matter of fact it is just $j^{(\nu\rho)}$, and together with the energy–momentum tensor of material fields it obeys a natural conservation equation for teleparallel manifolds. Some important consequences are obtained specially in the last section concerning the possibility of explaining the acceleration of the universe expansion without any need of a cosmological constant.

1 The energy–momentum tensor

We follow most of the conventions and notations of references [1] and [2]. However one difference is that we stick to consider the lagragian density of the gravitational field to be proportional to $+\overset{\circ}{R}$ (as it is usually accepted) instead of being proportional to $-\overset{\circ}{R}$ as is done in [1]. Any further difference will be made it explicit. Anyway let us at least remember that greek indices

are used for the coordinate holonomic quantities and latin indices are used for non–holonomic ones.

One way of viewing the kind of Riemannian spaces in which teleparallel theories are formulated is by taking as departure point the hypothesis that physics somehow establishes a canonical, smooth, path-independent isomorphism between the tangent spaces of any two points of the manifold and hence that we may take some orthonormal, but otherwise arbitrary reference basis, which we will call $\vec{u}_a$, and refer all vectors at all points to that basis (of course, we identify $\vec{u}_a$ with its preimage at any given point). Parallel transport can then be introduced as meaning to transport keeping constant components with respect to this basis, then Cartan covariant derivative is just the variation with respect to this reference basis expressed, for example, in terms of the coordinate basis. Mathematically, this has the consequence of accepting only parallelizable manifolds as physically meaningful, this is nothing more than just a topological condition on the sort of Riemannian spaces we deal with. In particular the coordinate vectors $\partial_\mu(x)$, no matter which sets of coordinates x we take, should be expressible in terms of the reference basis:

$$\partial_\mu(x) = h^a{}_\mu(x)\vec{u}_a \tag{1}$$

Let us also remember that the relationship between the Levi-Civita covariant derivative $\overset{\circ}{\nabla}$ due to the symmetric riemannian connection and the "Cartan" covariant derivative ∇ of the Weitzenböck connection is given by the difference between the Christoffel symbols of both covariant derivatives:

$$\Gamma^\rho{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu} + K^\rho{}_{\mu\nu} \tag{2}$$

Where $K^\nu{}_{\rho\nu}$ is the contorsion tensor given by:

$$K^\rho{}_{\mu\nu} = \frac{1}{2}\left(g^{\rho\alpha}\left[T_{\mu\alpha\nu} + T_{\nu\alpha\mu}\right] - T^\rho{}_{\mu\nu}\right) \tag{3}$$

Now, as a first point in the discussion we must clarify what can be considered a conservation equation for the energy–momentum tensor within teleparallel theory. Suppose we had some (symmetric) energy–momentum tensor $S_\mu{}^\nu$ and let w^μ be the components of some vector which is Cartan covariant constant. Such vectors do exist because any linear constant combination of vectors of the reference basis $\vec{u}_a$ is Cartan covariant constant. $S_\mu{}^\nu w^\mu$ represents the flow of the component of energy–momentum in the direction of the four-vector $\vec{w}$, so $\overset{\circ}{\nabla}_\nu(S_\mu{}^\nu w^\mu) = 0$ expresses the conservation of such $\vec{w}$ component. However we can write this as:

$$\begin{aligned} 0 &= \overset{\circ}{\nabla}_\nu(S_\mu{}^\nu w^\mu) = \nabla_\nu(S_\mu{}^\nu w^\mu) - S_\mu{}^\rho w^\mu K^\nu{}_{\rho\nu} \\ &= w^\mu\left(\nabla_\nu S_\mu{}^\nu - S_\mu{}^\rho T^\nu{}_{\nu\rho}\right) \end{aligned} \tag{4}$$

If we want conservation of energy–momentum in all directions, then it must hold:

$$\nabla_\nu S_\mu{}^\nu - S_\mu{}^\rho T^\nu{}_{\nu\rho} = 0 \tag{5}$$

This is not the same as the condition $\overset{\circ}{\nabla}_\nu S_\mu{}^\nu = 0$. If we substitute the Cartan covariant derivative by its classical counterpart, we reach another expression:

$$\overset{\circ}{\nabla}_\nu S_\mu{}^\nu - S_\sigma{}^\rho K^\sigma{}_{\mu\rho} = 0 \tag{6}$$

Of course, these equations reduce to the zero divergence condition in Minkowski spaces, however the important point is that (for second rank tensors in teleparallel spaces) they represent the correct generalization of the zero divergence condition of Minkowski space.

As it is well known Einstein's tensor $G^{\mu\nu}$ verifies $\overset{\circ}{\nabla}_\nu G^{\mu\nu} = 0$, so from Einstein's equation we do not get a conservation equation for the energy–momentum tensor of matter alone. For the energy–momentum tensor $\mathcal{T}$ of matter, accepting it to be a symmetric tensor, we rather get:

$$0 = \overset{\circ}{\nabla}_\nu \mathcal{T}_\mu{}^\nu \quad \Longrightarrow \quad \left(\nabla_\nu \mathcal{T}_\mu{}^\nu - \mathcal{T}_\mu{}^\rho T^\nu{}_{\nu\rho}\right) + \mathcal{T}_\sigma{}^\nu T^\sigma{}_{\nu\mu} = 0 \tag{7}$$

Comparing it with (5) we see it is not quite exactly the same, a further term is present. Let us remember that we are dealing with a lagrangian density Λ for the gravitational field which in teleparallel theories might in general be written as:

$$\Lambda = \kappa_g (a_1\Lambda_1 + a_2\Lambda_2 + a_3\Lambda_3) \tag{8}$$

where:

$$\Lambda_1 = g^{\lambda\mu} T^\alpha{}_{\alpha\lambda} T^\beta{}_{\beta\mu} \qquad \Lambda_2 = g^{\lambda\mu} T^\alpha{}_{\beta\lambda} T^\beta{}_{\alpha\mu} \tag{9}$$

$$\Lambda_3 = T^{\rho\beta\mu} T_{\rho\beta\mu} \qquad \kappa_g = \frac{c^4}{16\pi G} \tag{10}$$

The "default" values: $a_1 = 1$, $a_2 = -1/2$, $a_3 = -1/4$, produce a lagrangian density equivalent to general relativity, meaning that for those values the difference between $\overset{\circ}{R}$ and $(a_1\Lambda_1 + a_2\Lambda_2 + a_3\Lambda_3)$ is just a total divergence. As a matter of fact, adding $2g^{\alpha\beta}\overset{\circ}{\nabla}_\alpha T^\lambda{}_{\beta\lambda}$ one obtains $\overset{\circ}{R}$. However, for the moment, we are not going to fix the value of those coefficients, we want to keep open the possibility of choosing other values for them.

One way of obtaining the gravitational energy–momentum tensor is to directly reproduce the reasoning which in classical mechanics leads to the conservation of energy and write (we follow the standard work [4], which as a

matter of fact is just the first step of applying Noether's method considering the translational invariance of the lagrangian, see also [5], section 2):

$$\frac{\partial(\Lambda h)}{\partial x^\mu} = \frac{\partial(\Lambda h)}{\partial h^a{}_\nu}\frac{\partial h^a{}_\nu}{\partial x^\mu} + \frac{\partial(\Lambda h)}{\partial h^a{}_{\nu,\mu}}\frac{\partial h^a{}_{\nu,\mu}}{\partial x^\mu} \tag{11}$$

The field equations can be written as:

$$\frac{1}{h}h^a{}_\sigma\left[\frac{\partial(\Lambda h)}{\partial h^a{}_\nu} - \frac{\partial}{\partial x^\gamma}\left(h\frac{\partial\Lambda}{\partial h^a{}_{\nu,\gamma}}\right)\right] + \mathcal{T}_\sigma{}^\nu = 0 \tag{12}$$

And taking them into account one inmediately is led to:

$$\frac{1}{h}\frac{\partial}{\partial x^\gamma}\left[h\left(h^a{}_{\nu,\mu}\frac{\partial\Lambda}{\partial h^a{}_{\nu,\gamma}} - \Lambda\delta^\gamma_\mu\right)\right] - \mathcal{T}_\sigma{}^\nu\Gamma^\sigma{}_{\nu\mu} = 0 \tag{13}$$

One would like to identify the term within the round brackets in last equation with the energy–momentum tensor, but it is not a tensor, so the idea is to decompose it into tensorial and non–tensorial terms, hence we write previous equation as:

$$\frac{1}{h}\frac{\partial}{\partial x^\gamma}\left[h(Q_\mu{}^\gamma + N_\mu{}^\gamma)\right] - \mathcal{T}_\sigma{}^\nu\Gamma^\sigma{}_{\nu\mu} = 0 \tag{14}$$

Where $Q_\mu{}^\gamma$ is the tensorial part and $N_\mu{}^\gamma$ is a non–tensorial term. Expressing the partial derivative of the Q tensor as a Cartan covariant derivative one arrives inmediately to the following equation:

$$0 = \nabla_\gamma Q_\mu{}^\gamma - Q_\mu{}^\gamma T^\nu{}_{\nu\gamma} + \Gamma^\nu{}_{\nu\gamma}N_\mu{}^\gamma + Q_\sigma{}^\gamma\Gamma^\sigma{}_{\mu\gamma} + \frac{\partial N_\mu{}^\gamma}{\partial x^\gamma} - \mathcal{T}_\sigma{}^\nu\Gamma^\sigma{}_{\nu\mu} \tag{15}$$

Now the problem is to eliminate the non–tensorial terms of this equation. Of course, it must be possible to do so, because you cannot have an equalty between entities which transform in different ways (you may put the first two terms at one side and the other terms at the other side). The needed calculations for eliminating, or transforming, those non–tensorial terms are a bit cumbersome, but in the appendix it is indicated how it can be shown that by taking $Q_\mu{}^\gamma$ as:

$$\begin{aligned}
Q_\mu{}^\gamma &= \kappa_g\Big\{a_1\Big[2\Big(g^{\nu\gamma}T^\alpha{}_{\alpha\nu}T^\beta{}_{\beta\mu} - g^{u\beta}T^\alpha{}_{\alpha\nu}T^\gamma{}_{\beta\mu}\Big)\Big]\\
&+ a_2\Big[2\Big(g^{\nu\gamma}T^\alpha{}_{\beta\nu}T^\beta{}_{\alpha\mu} - g^{\nu\alpha}T^\gamma{}_{\beta\nu}T^\beta{}_{\alpha\mu}\Big)\Big]\\
&+ a_3\Big[4g^{\beta\gamma}g^{\alpha\nu}g_{\rho\tau}T^\rho{}_{\alpha\beta}T^\tau{}_{\nu\mu}\Big] - \Big[a_1\Lambda_1 + a_2\Lambda_2 + a_3\Lambda_3\Big]\delta^\gamma_\mu\Big\}
\end{aligned} \tag{16}$$

one arrives to the conclusion that

$$\Gamma^\nu{}_{\nu\gamma}N_\mu{}^\gamma + Q_\sigma{}^\gamma\Gamma^\sigma{}_{\mu\gamma} + \frac{\partial N_\mu{}^\gamma}{\partial x^\gamma} = \Gamma^\sigma{}_{\mu\nu}\mathcal{T}_\sigma{}^\nu \tag{17}$$

The expression for $Q_\mu{}^\gamma$ might seem strange at first sight, but it happens to be exactly $-j_\mu{}^\gamma$:

$$\frac{1}{h} h^a{}_\mu \frac{\partial(\Lambda h)}{\partial h^a{}_\gamma} = j_\mu{}^\gamma = -Q_\mu{}^\gamma \tag{18}$$

Hence we are led to the result:

$$\nabla_\gamma j_\mu{}^\gamma - j_\mu{}^\gamma T^\nu{}_{\nu\gamma} + \mathcal{T}_\sigma{}^\nu T^\sigma{}_{\mu\nu} = 0 \tag{19}$$

So taking into account both equations (19) and (7) one gets:

$$\nabla_\gamma(\mathcal{T}_\mu{}^\gamma + j_\mu{}^\gamma) - (\mathcal{T}_\mu{}^\gamma + j_\mu{}^\gamma)T^\nu{}_{\gamma\nu} = 0 \tag{20}$$

Which has exactly the form of equation (5) and so it can be interpreted as just expressing the conservation of total energy–momentum: the energy–momentum of the material fields $\mathcal{T}_\mu{}^\gamma$ plus the energy–momentum of the gravitational field $j_\mu{}^\gamma$. It also clarifies the meaning of the term $\mathcal{T}_\sigma{}^\nu T^\sigma{}_{\mu\nu}$. This term specifies the energy–momentum interchange between the gravitational field and the material one. It is the term which prevents energy–momentum of the gravitational field or energy–momentum of the material field from being conserved separately by themselves. The interpretation of $j_\mu{}^\gamma$ as the correct energy–momentum tensor for the gravitational field can be further underlined if one rewrites the field equations as:

$$\frac{1}{h} h^a{}_\sigma \frac{\partial}{\partial x^\gamma}\left(h \frac{\partial \Lambda}{\partial h^a{}_{\nu,\gamma}}\right) = j_\sigma{}^\nu + \mathcal{T}_\sigma{}^\nu \tag{21}$$

and compares it with the equations for the electromagnetic field in Minkowski space written as:

$$\frac{\partial}{\partial x^\gamma}\left(\frac{\partial \Lambda_e}{\partial A_{\nu,\gamma}}\right) = -\frac{J^\nu}{c} \tag{22}$$

Informally, it is usually accepted that this last equation says that currents are the sources of the electromagnetic field. Then, in the same sense, the previous equation might be interpreted as saying that energy–momentum of both the material fields and the gravitational field is the source of the gravitational field.

However, this is not the final word about the tensor we need, because $j_\sigma{}^\nu$ has the uncomfortable characteristic that in general it is not symmetric. If we accept the default values for the coefficients a_1, a_2, a_3 then there is an easy way out of the problem: Einstein's equations are a total of ten equations, however they do not completely determine the metric tensor. Diffeomorphisms comprise the gauge freedom in general relativity (see [6] pg. 438): any two solutions which are related by a diffeomorphism represent the same physical

solution. Now, if instead of considering the metric tensor as the final solution of a gravitational problem, we ask a complete solution of such a problem to be given by the specification of the sixteen functions $h^a{}_\sigma$ which give the coordinate basis vectors in terms of the arbitrary constant reference basis, then we have some further freedom, because once the metric tensor is given, we may have several "square roots" $h^a{}_\sigma$ which produce the same metric tensor: There are sixteen arbitrary $h^a{}_\sigma$ functions and only ten independent conditions imposed by the metric tensor. We have lots of "gauge freedom", so let us use part of that freedom to decree that the antisymmetric part of the energy–momentum tensor $j_\sigma{}^\nu$ should be zero. This "gauge condition" amounts just to a set of six equations, because in a four dimensional space an antisymmetric second rank tensor has only six independent components. So, although it is a crude way of counting degrees of freedom, we have increased by six the number of unknown functions when substituting the metric tensor as solution by the $h^a{}_\sigma$, but we have also added six additional equations, so we expect not to have changed the "gauge freedom". The gauge condition can be written as:

$$j^{[\gamma\nu]} = 0 = \left(g^{\mu\gamma} T^\nu{}_{\beta\mu} - g^{\mu\nu} T^\gamma{}_{\beta\mu}\right) g^{\lambda\beta} T^\alpha{}_{\alpha\lambda} - \frac{1}{2}\left(g^{\mu\gamma} T^\nu{}_{\beta\lambda} - g^{\mu\nu} T^\gamma{}_{\beta\lambda}\right) T^{\beta\lambda}{}_\mu \quad (23)$$

Of course it is a covariant condition: true in one coordinate system means true in all, so we are not limiting the set of coordinates in which the theory is formulated: we are not imposing conditions on the sort of diffeomorphisms which might be used. As a matter of fact, it has previously been argued that teleparallel theories may have too much gauge freedom (see [7], [8]) and that they suffer from a problem of non–predictability of torsion. Although a formal proof should be investigated, we clearly expect this gauge condition to fix such problems. At least, the introduction of this condition invalidates the reasoning supporting such assertions, because clearly these additional six equations have not been taken into account when studying the predictability of torsion. Furthermore: being an algebraic condition on the torsion tensor, not every boundary condition is acceptable, because the boundary condition must also obey the gauge condition.

Accepting such a gauge condition, the energy–momentum of the gravitational field turns out to be symmetric, which just means that $j^{\gamma\nu} = j^{(\gamma\nu)}$. And furthermore it is inmediate to check that it has zero trace. Needless to say it is a perfectly covariant local definition of energy–momentum for the gravitational field.

There is one further point which merits some comment. Teleparallel theories have some degree of freedom in the way the coefficients are chosen. However it is only for the case in which we obtain the teleparallel equivalent to general relativity when the resulting equations are symmetric (we obtain Ein-

stein's tensor). So it is only in this case in which we have the freedom to impose that the energy–momentum tensor of the gravitational field should be symmetric (as there is no other possibility in empty space). So this condition eliminates the rest of possibilities for the coefficients.

We may even generalize: if we restrict teleparallel theories to just this case, as it has been said before, we obtain symmetric equations (Einstein's tensor), so what would happen if the energy–momentum tensor of a material field were not symmetric?. The most natural answer in this hypothetical case would be to change the gauge condition so that the antisymmetric part of the energy–momentum tensor of the gravitational field just cancels the antisymmetric part of the energy–momentum tensor of the material fields, and only the symmetric part plays a role in Einstein's equation.

2 The energy content of homogeneous, isotropic universes.

The energy content of homogeneous, isotropic universes has already been computed in other papers (see [3], for example). However, although our results are quite similar, there are some points which should be noted: previous works have used pseudo–tensors and of course have never taken into account the gauge condition. Usually they need to integrate over a space section. Given that we are dealing with a isotropic, homogeneous universe, if we were really working with a true gravitational energy density, one would expect the energy density to be homogeneous, and if the total energy turns out to be zero, it is quite unintuitive that it is not zero at every point. Of course, one answer is that usually authors are not working with the true energy density of the gravitational field.

To calculate the energy content of the three cases of null, positive and negative curvature, we must first find "square roots" of the respective metrics. Not every square root is acceptable, we must also impose the gauge condition: the energy–momentum tensor derived from them should be symmetric. However for these problems we must not only obtain symmetrical tensors, but also diagonal, otherwise we would have some preferred direction in space. It must be noted that most calculations in this section have been done with Maple© 9.5 running on Ubuntu Linux.

The easiest case is the one of zero curvature. We use a "cartesian" coordinate system with coordinates ct, x, y, z, and we postulate the following matrix of gravitational potential vectors (the role played by the coordinate vectors is

similar to that of vector potentials):

$$h^a{}_\alpha = \mathrm{diag}\,(1, a(t), a(t), a(t)) \tag{24}$$

Using this potentials, the metric is just the very well known diagonal metric of flat space $g_{\nu\mu} = \mathrm{diag}(1, -a^2(t), -a^2(t), -a^2(t))$. The energy–impulse tensor for such a space is:

$$j_0^0 = -6\kappa_g \left(\frac{\dot{a}(t)}{a(t)}\right)^2 \qquad j_1^1 = j_2^2 = j_3^3 = 2\kappa_g \left(\frac{\dot{a}(t)}{a(t)}\right)^2 \tag{25}$$

Where the dot signals ordinary differentiation with respect to ct. The first point which deserves attention is that energy density is negative, so it seems that there is at least a known field whose energy density takes negative values. It is also purelly "kinetical" in this case: it is proportional to the square of the speed at which $a(t)$ changes, and the minus sign tells us that absortion of (positive) energy will decrease this speed.

Let us consider first the case of a dust-filled universe. We know that for such a case $a(t) = C_d t^{2/3}$, so the gravitational energy density is proportional to $-t^{-2}$ which, when multiplied by $a^3 \propto t^2$ to take into account the increase in volume, just gives constant energy: dust does not contribute to any variation of energy of the gravitational field, it does not interchange energy with the gravitational field.

Consider now the case of a universe filled with just radiation, being ρ its energy density. It is well-known that in such a case the solution for $a(t)$ is of the form $a(t) = C_r t^{1/2}$. So the gravitational energy density is also proportional to $-t^{-2}$ (it is the square of a logarithmic derivative, so no matter the exponent it will be proportional to $-t^{-2}$), which when multiplied by $a^3(t)$, to take into account the increase of volume, gives the result that energy of gravitational field changes as $-t^{-1/2}$, which is an increase and which is just the rate needed to compensate the rate at which energy of radiation decreases: ρa^4 is constant, so ρa^3 decreases as $a^{-1} \propto t^{-1/2}$. The absortion of positive energy from radiation just decreases the rate at which universe expands. In the dust-filled case, the decrease in speed is just to compensate the increase in volume, so that total energy is the same. As energy of light is absorbed by the gravitational field, its "kinetic" energy increases (it decreases its "speed"). A radiation dominated universe expands at a slower rate ($t^{1/2}$) than a dust filled one ($t^{2/3}$): absortion of energy decreases its speed.

Even more clear, for the flat universe Einstein's equation can be used to calculate the energy–momentum tensor of matter:

$$\mathcal{T}_0^0 = 6\kappa_g \left(\frac{\dot{a}}{a}\right)^2 \qquad \mathcal{T}_1^1 = \mathcal{T}_2^2 = \mathcal{T}_3^3 = 2\kappa_g \left(\frac{\dot{a}^2 + 2\ddot{a}a}{a^2}\right) \tag{26}$$

Looking at the first term we see that the energy density of the matter fields is just the same (but positive) as the energy density of the gravitational field, so that total energy density is zero. This is a fact which also happens in the case of positive curvature.

For positive curvature universe the way to get an acceptable square root of the metric is to remember that S^3 is parallelizable, so it is easy to get three ortonormal vectors which are tangent to it and then take them as the spatial part of the reference basis at each point. Of course, by an acceptable square root we mean one which renders a symmetric gravitational energy–momentum tensor. A diagonal square root, as it is usually taken, does not. The previous idea can be summed up by saying that we postulate an ${h^a}_\alpha$ matrix given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t)c(\Theta) & -a(t)s(\Psi)c(\Psi)s(\Theta) & -a(t)s^2(\Psi)s^2(\Theta) \\ 0 & a(t)s(\Theta)c(\Phi) & a(t)s(\Psi)\left(s(\Psi)s(\Phi)+c(\Psi)c(\Theta)c(\Phi)\right) & a(t)s(\Psi)s(\Theta)\left(s(\Psi)c(\Theta)c(\Phi)-c(\Psi)s(\Phi)\right) \\ 0 & a(t)s(\Theta)s(\Phi) & a(t)s(\Psi)\left(c(\Psi)c(\Theta)s(\Phi)-s(\Psi)c(\Phi)\right) & a(t)s(\Psi)s(\Theta)\left(c(\Psi)c(\Phi)+s(\Psi)c(\Theta)s(\Phi)\right) \end{pmatrix} \tag{27}$$

Where $s \equiv \sin$ and $c \equiv \cos$. This matrix leads to the spherical metric:

$$g_{\nu\mu} = \mathrm{diag}(1, -a^2(t), -a^2(t)s^2(\Psi), -a^2(t)s^2(\Psi)s^2(\Theta)) \tag{28}$$

The energy–momentum tensor of matter is given by:

$$\mathcal{T}_0^0 = 6\kappa_g \frac{\dot{a}^2+1}{a^2} \qquad \mathcal{T}_1^1 = \mathcal{T}_2^2 = \mathcal{T}_3^3 = 2\kappa_g \frac{2\ddot{a}a+\dot{a}^2+1}{a^2} \tag{29}$$

The gravitational energy–momentum tensor is given by:

$$j_0^0 = -6\kappa_g \frac{\dot{a}^2+1}{a^2} \qquad j_1^1 = j_2^2 = j_3^3 = 2\kappa_g \frac{\dot{a}^2+1}{a^2} \tag{30}$$

So we get into the same situation in which the total energy is zero.

It is a bit more difficult to find an acceptable square root for the negative curvature case. As a matter of fact, in the acceptable solution which has been found, the potential vectors cannot be so neatly separated into spatial and temporal parts, however they do lead to the correct metric and to a diagonal energy–momentum tensor. The solution found for matrix ${h^a}_\alpha$ is:

$$\begin{pmatrix} ch(\Psi) & a(t)sh(\Psi) & 0 & 0 \\ sh(\Psi)c(\Theta) & a(t)ch(\Psi)c(\Theta) & -a(t)sh(\Psi)s(\Theta) & 0 \\ sh(\Psi)s(\Theta)c(\Phi) & a(t)ch(\Psi)s(\Theta)c(\Phi) & a(t)sh(\Psi)c(\Theta)c(\Phi) & -a(t)sh(\Psi)s(\Theta)s(\Phi) \\ sh(\Psi)s(\Theta)s(\Phi) & a(t)ch(\Psi)s(\Theta)s(\Phi) & a(t)sh(\Psi)c(\Theta)s(\Phi) & a(t)sh(\Psi)s(\Theta)c(\Phi) \end{pmatrix} \tag{31}$$

Where $ch \equiv \cosh$ and $sh \equiv \sinh$. It is inmediate to check that the hyperbolic metric is obtained with these potentials:

$$g_{\nu\mu} = \mathrm{diag}(1, -a^2(t), -a^2(t)\sinh^2(\Psi), -a^2(t)\sinh^2(\Psi)\sin^2(\Theta)) \tag{32}$$

The energy–momentum tensor of matter is given by:

$$\mathcal{T}_0^0 = 6\kappa_g \frac{\dot{a}^2 - 1}{a^2}; \qquad \mathcal{T}_1^1 = \mathcal{T}_2^2 = \mathcal{T}_3^3 = 2\kappa_g \frac{2\ddot{a}a + \dot{a}^2 - 1}{a^2}; \tag{33}$$

While the gravitational energy–momentum tensor is given by:

$$j_0^0 = -6\kappa_g \frac{(\dot{a} - 1)^2}{a^2} \qquad j_1^1 = j_2^2 = j_3^3 = 2\kappa_g \frac{(\dot{a} - 1)^2}{a^2} \tag{34}$$

But we do not reach the same conclusion: the total energy is not zero. This is somewhat unexpected: if the total energy were zero in the three models, it will agree quite well with the idea of originating from a state of zero total energy. However this case is different and so, unless some other interpretation is found, it raises the question of whether, although mathematically possible, it is physically reasonable: why should universe begin in a state of energy different from zero?. On the other hand, a flat universe has the minimum matter density compatible with zero total energy.

3 Last comments.

We have already argued that equation (5) expresses the conservation of energy–impulse, so let us write that equation as:

$$\diamond_\nu S_\mu{}^\nu \equiv \nabla_\nu S_\mu{}^\nu - S_\mu{}^\rho T^\nu{}_{\nu\rho} = 0 \tag{35}$$

We have also seen in equation (7) that Einstein's equation implies:

$$\diamond_\nu \mathcal{T}_\mu{}^\nu = -\mathcal{T}_\sigma{}^\nu T^\sigma{}_{\nu\mu} \tag{36}$$

Where $\mathcal{T}$ is the energy–momentum tensor of matter fields. Let us suppose we are dealing with a perfect fluid in an isotropic homogeneous universe. Let us consider for example the case of flat universe. The energy–momentum tensor can be written in such case as:

$$\mathcal{T}_{\mu\nu} = \mathrm{diag}\left(\rho, pa^2, pa^2, pa^2\right) \tag{37}$$

Where $\rho = \rho(t)$ is the mass-energy density, $p = p(t)$ the pressure and the $a^2(t)$ factors come from the metric. The right hand of equation (36) can be easily computed and turns out to be:

$$\left(-3\frac{\dot{a}}{a}p(t), 0, 0, 0\right) \tag{38}$$

The first thing which stands out is that if $p(t) \neq 0$ then in an expanding universe, mass–energy of the material field (by itself) is not conserved. We have seen such a behaviour when we considered the radiation–filled universe before: a positive pressure means that the gravitational field drains the positive energy from the "material" field. As a matter of fact we have also seen the case $p = 0$ in the dust-filled universe and there was no energy interchange. Let us turn to the other possible case. Suppose there is some "spontaneous matter emission" process, then if matter is created from the gravitational field, pressure must be negative. Of course, we do not know what exactly to put in the left hand, and the matter–emission process must have very low probability of ocurrence because otherwise it would have already been detected. Let us just put a small "constant" λ in the left hand side of equation (36), mainly because we have no better guess, then we may write:

$$3p(t) = -\lambda \frac{a}{\dot{a}} \tag{39}$$

So although the process may have very low probability of ocurrence, the negative pressure increases with the expansion of the universe. It may of course overcome the mass term in the equation which determines the acceleration of the expansion of the universe (see for example [6] pg. 97):

$$3\frac{\ddot{a}}{a} = -\frac{1}{4\kappa_g}\left[\rho(t) + 3p(t)\right] \tag{40}$$

From that moment, positive acceleration sets in. We do not need to have a cosmological constant to explain it. In fact we cannot consider this λ as a constant, it may depend on the strength of the gravitational field, and also even in case we accept the possibility of matter–emission processes, their rate must compensate the energy absortion rate of the gravitational field from electromagnetic radiation. We would need a quantum theory of gravitation to be able to calculate $\lambda(t)$. However, we may get an idea of the order of magnitude of λ by considering zero the acceleration, taking the Hubble constant $H_0 = \dot{a}/a$ to be 70 (km/s)/Mpc, and taking the density of the universe to be the critical one $\approx 2 \times 10^{-26}\mathrm{kg/m^3}$. We get $\lambda \approx 5 \times 10^{-44}\mathrm{kg/(m^3 s)}$. This is the order of magnitude of the rate at which matter is created at the expense of the gravitational field (of course, it says nothing about what sort of particles are created). There is nothing to prevent gravitational field from falling even further down in energy levels (as it is the usual objection to negative energies), only that the rate is extremely slow. Supposedly, a quantum theory of gravitation should be able to explain this rate. Anyway it sets an experimental test for any such a theory: see if within lowest order perturbation you can obtain something similar.

A The calculation of Q.

First place we will need the explicit form of Einstein equations written in terms of the torsion tensor and metric tensors. This has been done quite a number of times, and we only put explicitly and directly in terms of these tensor. After working out the calculations implicit in equations (12), Einstein's equations can be written as:

$$
\begin{aligned}
-\frac{1}{\kappa_g}\mathcal{T}_\sigma{}^\nu = {} & a_1\left[2g^{\lambda\nu}\nabla_\sigma T^\alpha{}_{\lambda\alpha} + \delta^\nu_\sigma(2\Lambda_0 - \Lambda_1)\right] + a_2\left[2\nabla_\gamma T^{\gamma\nu}{}_\sigma - 2\nabla_\gamma T^{\nu\gamma}{}_\sigma\right. \\
& \left. - 2g^{\lambda\nu}T^\alpha{}_{\beta\lambda}T^\beta{}_{\alpha\sigma} + 2g^{\lambda\nu}T^\gamma{}_{\gamma\rho}T^\rho{}_{\sigma\lambda} - 2g^{\lambda\rho}T^\gamma{}_{\gamma\rho}T^\nu{}_{\sigma\lambda} + \Lambda_2\delta^\nu_\sigma\right] + \\
& + a_3\left[-4T^{\rho\beta\nu}T_{\rho\beta\sigma} - 4T^\gamma{}_{\gamma\rho}T_\sigma{}^{\nu\rho} + 2T^{\nu\alpha\lambda}T_{\sigma\alpha\lambda} - 4g_{\sigma\tau}\nabla_\gamma T^{\tau\gamma\nu} + \Lambda_3\delta^\nu_\sigma\right] \qquad (41)
\end{aligned}
$$

To accomplish the elimination of nontensorial elements in equation (15) let us do the needed calculations for each of the three parts of the lagrangian density. For $\Lambda_1 = g^{\lambda\mu}T^\alpha{}_{\alpha\lambda}T^\beta{}_{\beta\mu}$ one has that:

$$h^a{}_{\nu,\mu}\frac{\partial\Lambda_1}{\partial h^a{}_{\nu,\gamma}} - \Lambda_1\delta^\gamma_\mu = -2\left(g^{\lambda\gamma}T^\alpha{}_{\alpha\lambda}\Gamma^\beta{}_{\beta\mu} - g^{\lambda\beta}T^\alpha{}_{\alpha\lambda}\Gamma^\gamma{}_{\beta\mu}\right) - \Lambda_1\delta^\gamma_\mu \qquad (42)$$

And one may take:

$$\frac{{}_1Q_\mu{}^\gamma}{\kappa_g} = 2\left(g^{\lambda\gamma}T^\alpha{}_{\alpha\lambda}T^\beta{}_{\beta\mu} - T^\alpha{}_{\alpha\lambda}T^{\gamma\lambda}{}_\mu\right) - \Lambda_1\delta^\gamma_\mu \qquad (43)$$

and

$$\frac{{}_1N_\mu{}^\gamma}{\kappa_g} = -2\left(g^{\lambda\gamma}T^\alpha{}_{\alpha\lambda}\Gamma^\beta{}_{\mu\beta} - g^{\lambda\beta}T^\alpha{}_{\alpha\lambda}\Gamma^\gamma{}_{\mu\beta}\right) \qquad (44)$$

The subindex 1 of, for example, ${}_1N^\gamma_\mu$ just refers to the fact that the term comes from Λ_1, it is not any sort of spatial index. So after some work, which may involve using the condition that the curvature of the Weitzenböck connection is zero for simplifying some expressions, one finds that:

$$
\begin{aligned}
& \frac{1}{\kappa_g}\left[{}_1N_\mu{}^\gamma\Gamma^\nu{}_{\nu\gamma} + {}_1Q_\sigma{}^\gamma\Gamma^\sigma{}_{\mu\gamma} + \frac{\partial({}_1N_\mu{}^\gamma)}{\partial x^\gamma}\right] = \\
& -\left(2g^{\lambda\nu}\nabla_\sigma T^\alpha{}_{\lambda\alpha} + \delta^\nu_\sigma(2\Lambda_0 - \Lambda_1)\right)\Gamma^\sigma{}_{\mu\nu} \qquad (45)
\end{aligned}
$$

For $\Lambda_2 = g^{\lambda\mu}T^\alpha{}_{\beta\lambda}T^\beta{}_{\alpha\mu}$ one gets:

$$h^a{}_{\nu,\mu}\frac{\partial\Lambda_2}{\partial h^a{}_{\nu,\gamma}} - \Lambda_2\delta^\gamma_\mu = -2\left(g^{\lambda\gamma}T^\alpha{}_{\beta\lambda}\Gamma^\beta{}_{\alpha\mu} - g^{\lambda\kappa}T^\gamma{}_{\beta\lambda}\Gamma^\beta{}_{\kappa\mu}\right) - \Lambda_2\delta^\gamma_\mu \qquad (46)$$

And then one may take:

$$\frac{{}_2Q_\mu{}^\gamma}{\kappa_g} = 2\left(g^{\lambda\gamma}T^\alpha{}_{\beta\lambda}T^\beta{}_{\alpha\,\mu} - T^\gamma{}_{\beta\lambda}T^{\beta\lambda}{}_\mu\right) - \Lambda_2\delta^\gamma_\mu \tag{47}$$

and

$$\frac{{}_2N_\mu{}^\gamma}{\kappa_g} = -2\left(g^{\lambda\gamma}T^\alpha{}_{\beta\,\lambda}\Gamma^\beta{}_{\mu\alpha} - g^{\lambda\kappa}T^\gamma{}_{\beta\lambda}\Gamma^\beta{}_{\mu\kappa}\right) \tag{48}$$

So after some second work, one finds that:

$$\frac{1}{\kappa_g}\left[{}_2N_\mu{}^\gamma\Gamma^\nu{}_{\nu\gamma} + {}_2Q_\sigma{}^\gamma\Gamma^\sigma{}_{\mu\,\gamma} + \frac{\partial({}_2N_\mu{}^\gamma)}{\partial x^\gamma}\right] = -2\Gamma^\sigma{}_{\mu\nu}\Big(\nabla_\gamma T^{\gamma\nu}{}_\sigma - \nabla_\gamma T^{\nu\gamma}{}_\sigma$$
$$-g^{\lambda\nu}T^\alpha{}_{\beta\lambda}T^\beta{}_{\alpha\sigma} + g^{\lambda\nu}T^\gamma{}_{\gamma\rho}T^\rho{}_{\sigma\lambda} - g^{\lambda\rho}T^\gamma{}_{\gamma\rho}T^\nu{}_{\sigma\lambda} + \Lambda_2\delta^\nu_\sigma\Big) \tag{49}$$

Finally for $\Lambda_3 = g^{\lambda\mu}g^{\alpha\beta}g_{\rho\sigma}T^\rho{}_{\alpha\lambda}T^\sigma{}_{\beta\mu}$ one has:

$$h^a{}_{\nu,\mu}\frac{\partial\Lambda_3}{\partial h^a{}_{\nu,\gamma}} - \Lambda_3\delta^\gamma_\mu = -4g^{\lambda\,\gamma}g^{\alpha\beta}g_{\rho\sigma}T^\rho{}_{\alpha\lambda}\Gamma^\sigma{}_{\beta\mu} - \Lambda_3\delta^\gamma_\mu \tag{50}$$

And then one may take:

$$\frac{{}_3Q_\mu{}^\gamma}{\kappa_g} = 4g^{\lambda\,\gamma}g^{\alpha\beta}g_{\rho\sigma}T^\rho{}_{\alpha\lambda}T^\sigma{}_{\beta\mu} - \Lambda_3\delta^\gamma_\mu \tag{51}$$

and

$$\frac{{}_3N_\mu{}^\gamma}{\kappa_g} = -4g^{\lambda\,\gamma}g^{\alpha\beta}g_{\rho\sigma}T^\rho{}_{\alpha\lambda}\Gamma^\sigma{}_{\mu\beta} \tag{52}$$

So after some third work, one finds that:

$$\frac{1}{\kappa_g}\left[{}_3N_\mu{}^\gamma\Gamma^\nu{}_{\nu\gamma} + {}_3Q_\sigma{}^\gamma\Gamma^\sigma{}_{\mu\,\gamma} + \frac{\partial({}_3N_\mu{}^\gamma)}{\partial x^\gamma}\right] =$$
$$-4\Gamma^\sigma{}_{\mu\nu}\left(-T^{\rho\beta\nu}T_{\rho\beta\sigma} - T^\gamma{}_{\gamma\rho}T_\sigma{}^{\nu\rho} + \frac{1}{2}T^{\nu\alpha\lambda}T_{\sigma\alpha\lambda} - \nabla_\gamma T_\sigma{}^{\gamma\nu} + \Lambda_3\delta^\nu_\sigma\right) \tag{53}$$

So taking into account Einstein's equation (41), one arrives to the conclusion that:

$$N_\mu{}^\gamma\Gamma^\nu{}_{\nu\gamma} + Q_\sigma{}^\gamma\Gamma^\sigma{}_{\mu\,\gamma} + \frac{\partial(N_\mu{}^\gamma)}{\partial x^\gamma} = \Gamma^\sigma{}_{\mu\nu}\mathcal{T}_\sigma{}^\nu \tag{54}$$

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